Similar and Congruent Figures

Warmup

Solve each proportion.

a) $\frac{x}{6} = \frac{1}{2}$ **b)** $\frac{3}{y} = \frac{2}{3}$

c) $\frac{a}{2} = \frac{2}{a}$ **d)** $\frac{x+2}{2} = \frac{4}{x}$

a) $\frac{x}{6} = \frac{1}{2}$

$6\left[\frac{x}{6}\right] = \left[\frac{1}{2}\right]6$

$\frac{\cancel{6}x}{\cancel{6}} = \frac{6}{2}$

$x = 3$

b) $\frac{3}{y} = \frac{2}{3}$

$y\left[\frac{3}{y}\right] = \left[\frac{2}{3}\right]y$

$\frac{3\cancel{y}}{\cancel{y}} = \frac{2y}{3}$

$3 = \frac{2y}{3}$

$3\left[3\right] = \left[\frac{2y}{3}\right]3$

$9 = \frac{6y}{3}$

$9 = 2y$

$\frac{9}{2} = \frac{2y}{2}$

$\frac{9}{2} = y$

$4.5 = y$

c) $\frac{a}{2} = \frac{2}{a}$

$2\left[\frac{a}{2}\right] = \left[\frac{2}{a}\right]2$

$\frac{\cancel{2}a}{\cancel{2}} = \frac{4}{a}$

$a = \frac{4}{a}$

$a\left[a\right] = \left[\frac{4}{a}\right]a$

$a^2 = \frac{4\cancel{a}}{\cancel{a}}$

$a^2 = 4$

$\sqrt{a^2} = \sqrt{4}$

$a = 2$ or $a = -2$

eg $(2)^2 = 4$
and
$(-2)^2 = 4$

d) $\frac{x+2}{2} = \frac{4}{x}$

$2\left[\frac{x+2}{2}\right] = \left[\frac{4}{x}\right]2$

$\frac{\cancel{2}(x+2)}{\cancel{2}} = \frac{8}{x}$

$x + 2 = \frac{8}{x}$

$x\left[x+2\right] = \left[\frac{8}{x}\right]x$

$x^2 + 2x = 8$

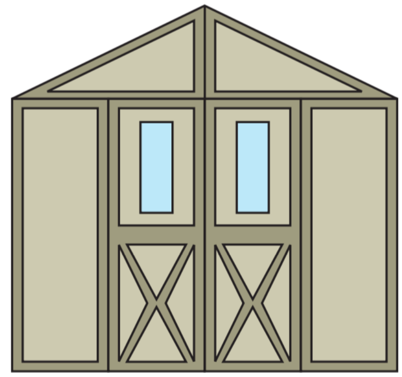
$x^2 + 2x - 8 = 8 - 8$

$x^2 + 2x - 8 = 0$

$(x-2)(x+4) = 0$

$\therefore x = 2$ or $x = -4$

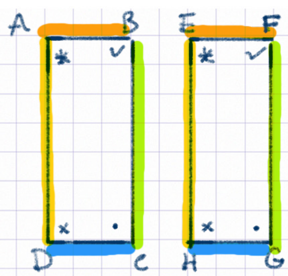
Example 1



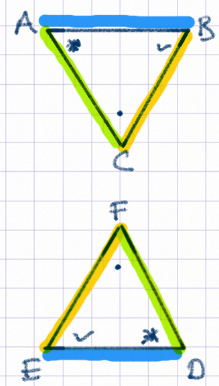
How can we recognize similar and congruent figures?

a. Look at the front face of the shed.

   i. What pairs of congruent figures can you find?
   ii. Copy two or three pairs of congruent figures from the drawing into the space below.
   iii. For each pair of congruent figures you drew, label the vertices.
   What special properties do the corresponding sides and angles have?



equal sides	equal angles
AB = EF	∠ABC = ∠EFG
BC = FG	∠BCD = ∠FGH
CD = GH	∠CDA = ∠GHE
DA = HE	∠DAB = ∠HEF



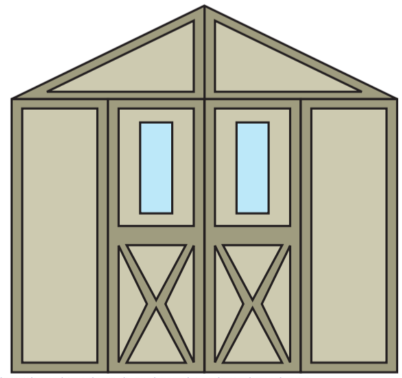
equal sides	equal angles
AB = DE	∠ABC = ∠DEF
BC = EF	∠BCA = ∠EFD
CA = FD	∠CAB = ∠FDE

b. Review your work from part a).

   Now, summarize: what are the properties of congruent figures?

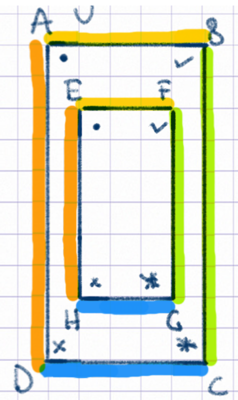
It appears that congruent figures always have corresponding sides that are all equal and corresponding angles that are all equal.

c. Look again at the front face of the shed.



   i. What pairs of similar, but not congruent, figures can you find?
   ii. Copy a pair of similar figures from the drawing into the space below.
   iii. Label the vertices of the figures. What special properties do the corresponding sides and angles have?

c)



So... the windows (light blue) enclosed in the shed door appear "similar".

Notice that... corresponding angles are equal (see symbols used).

Corresponding sides are scaled up (or down depending on which "direction" you compare the side lengths in.

corresponding side lengths are proportional

eg. AB = 5 units
EF = 3 units
∴ AB : EF = 5 : 3

eg BC = 10 units
FG = 6 units
∴ BC : FG
= 10 : 6
= 5 : 3
BC is 66% larger than FG

d. Complete the table below to consolidate your thoughts.

   i. What is true about the corresponding angles in two congruent figures? In two similar figures?
   ii. What is true about the corresponding side lengths in two congruent figures? In two similar figures?

	corresponding angles are...	**corresponding side lengths are...**
In congruent figures	all equal	all equal
In similar figures	all equal	proportional (have ratios that are all equal)

Example 2

If two triangles are congruent, what does that mean?

- Three corresponding side lengths are equal

- Three corresponding angles are equal

In short, there are:

six conditions to show the congruence of two triangles

Is it possible to prove the congruence of two triangles without meeting all of these conditions?

Shortforms are sometimes used to describe congruency conditions between triangles. Below:

- S means "side"
- A means "angle"
- H means "hypotenuse"

Here are five possible congruency conditions between two triangles:

i. SSS ii. SAS iii. ASA iv. AAS v. HS

So the first condition, SSS, suggests that two triangles are congruent if we know only that three corresponding sides are of equal length and know nothing of the angles. Is this true? Why?

The second condition, SAS, suggests that two triangles are congruent if we know only that two corresponding sides are of equal length with one corresponding equal angle "sandwiched between" the equal sides. We know nothing else about the triangles. Can this be true? How?

On the following pages, use diagrams and brief written explanations to determine, through your own reasoning, whether all five of these "congruency conditions" are in fact true.

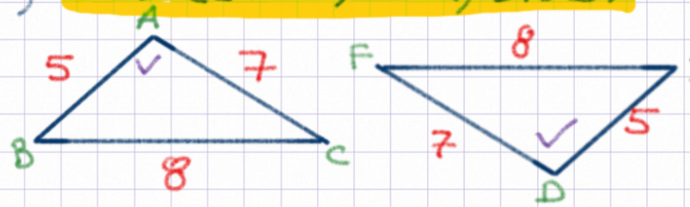
★ to deduce is to build an argument on previously established facts.

Two triangles must always meet all six conditions to be congruent.

That is: · 3 corresponding sides of equal length
· 3 corresponding angles that are equa

However... if we are not given all 6 pieces of information, we can
★ deduce whether the other angles and sides are equal.

i) SSS (side, side, side).



① yes, triangles must be congruent

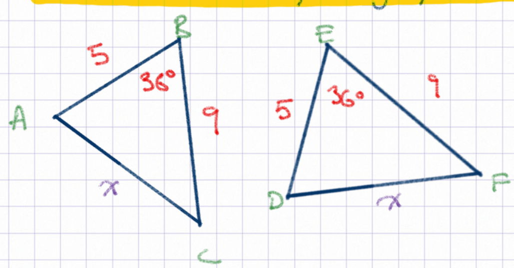
② ∠A is between sides AB and AC of lengths 5 and 7, joined by a side BC of length 8

③ same as ② applies for ∠D
∴ ∠A = ∠D, shown by ✓

④ same logic applies for other angles, ∠B = ∠E and ∠C = ∠F

⑤ ∴ ABC is congruent to DEF.

ii) SAS (side, angle, side).



① Angle B at 36° is between sides AB (length of 5) and BC (length of 9). Side AC joins AB and BC.

② The same conditions are true in △DEF.

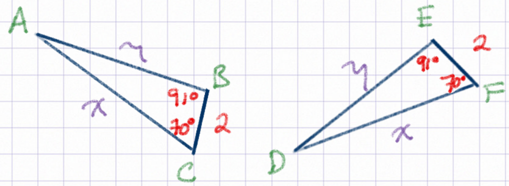
③ ∴ AC = DF (see x).

④ We don't know what x is but the third side is x units long in both triangles.

⑤ Now we are back to SSS (three known sides).

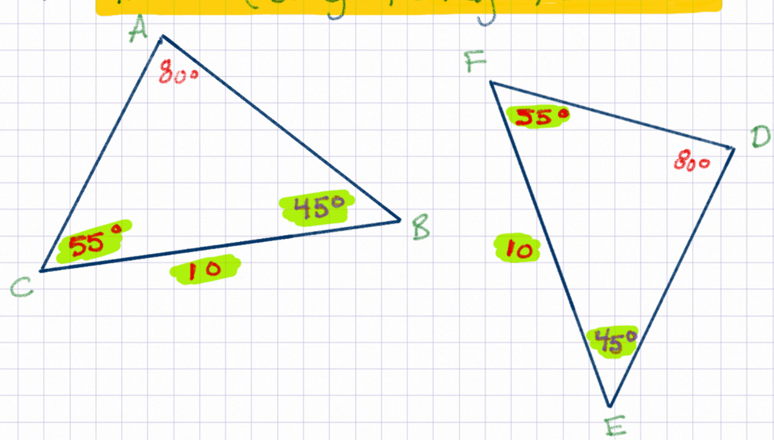
⑥ ∴ ABC must be congruent to DEF.

iii) ASA (angle, side, angle).



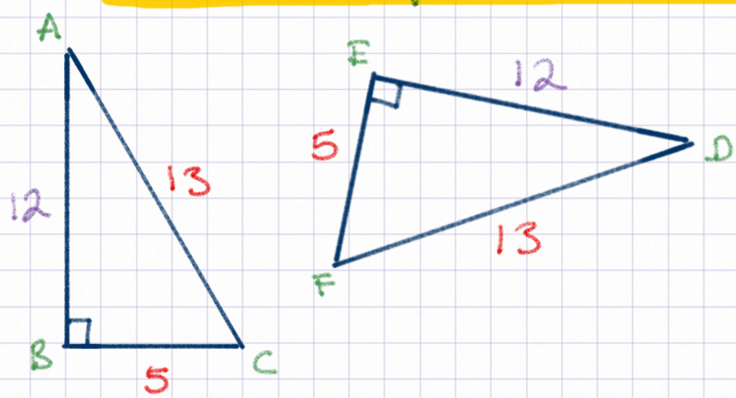
① CA extends from BC at a 70° angle.
BA " " " " " 91° " .
CA and BA are joined by BC of length 2.
② FD extends from EF at a 70° angle.
ED " " " " " 91° angle.
FD and ED are joined by EF of length 2.
③ ∴ CA = FD = x must be true.
④ ∴ BA = ED = y " " " .
⑤ we don't know x and y but we know sides of those lengths must exist in both triangles.
⑥ Now we are back at SSS.
△ABC must be congruent to △DEF.

iv) AAS (angle, angle, side).



① ∠B must be 180 − 80° − 55 = 45°
② ∠E " " " " " " .
③ ∴ ∠B = ∠E = 45° (added above in purple)
④ Now we are back to ASA.
⑤ ∴ △ABC must be congruent to △DEF.

v) HS (hypotenuse-side).



① If we have a hypotenuse we must have the 90° angles shown.

② By the Pythagorean Theorem, AB = ED = 12

$$hyp^2 = leg_1^2 + leg_2^2$$
$$(13)^2 = (5)^2 + x^2$$
$$169 = 25 + x^2$$
$$-25 \quad -25$$
$$144 = x^2$$
$$\sqrt{144} = \sqrt{x^2}$$
$$12 = x$$

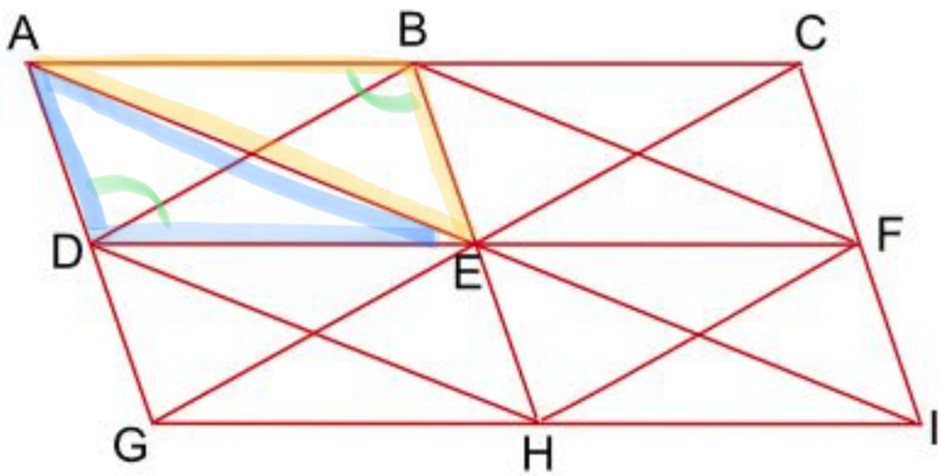
shown in purple on diagram.

③ Now we are back to SSS.

$\therefore \triangle ABC$ must be congruent to $\triangle DEF$.

Opportunity to Learn

1. How many triangles congruent to triangle ABE (including itself) are there in the following diagram?



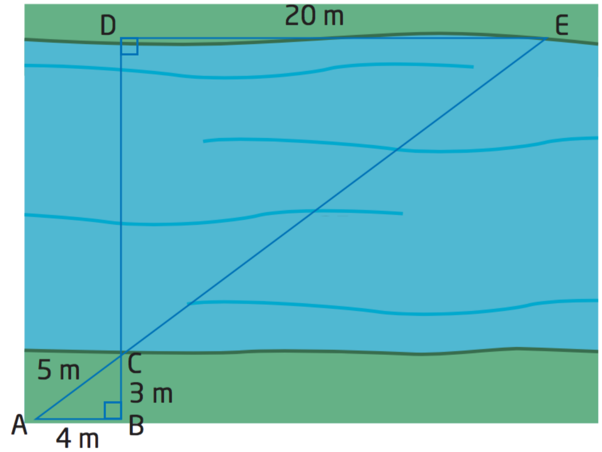
- $\triangle ABE$ has one obtuse angle ($\angle ABE$) and two acute angles ($\angle EAB$ and $\angle BEA$).
- In the upper left portion of the figure, only $\triangle EDA$ also has that same obtuse angle with two acute angles.
  - ↳ $\triangle EDA \cong \triangle ABE$
  - "is congruent to"
- $\triangle BDE$ and $\triangle BDA$ both have three acute angles and so they cannot be congruent to $\triangle ABE$.
- This same logic applies to the upper right and lower left and lower right portions of the figure.
- So in total, there are 8 triangles congruent to $\triangle ABE$ (including $\triangle ABE$) in this figure.

2. Surveyors have staked out the following points on opposite banks of a river.

   a) Are triangles ABC and EDC congruent? Are they similar?

   b) If the surveyors want to build a footbridge from points C to D, how long will it need to be?

   c) If the surveyors want to build a footbridge from points C to E, how long will it need to be?



a) $\triangle ABC$ cannot be congruent to $\triangle EDC$ because $AB \neq DE$.

Are the triangles similar? Hmm...

Well $\angle B = \angle D = 90°$ (given)

$\angle ACB = \angle ECD$ (by OAT from grade 9, the opposite angle theorem).

$\therefore \angle E = \angle A$ (by SATT, angles in a $\triangle$ have a sum of $180°$).

If there are three equal angles, $\triangle ABC$ must be proportional to $\triangle EDC$ (in other words, $\triangle EDC$ is a scaled up version of $\triangle ABC$).

b) $\frac{AB}{ED} = \frac{4}{20}$

$= \frac{1}{5}$

$\therefore \frac{BC}{DC} = \frac{1}{5}$

$\frac{3}{x} = \frac{1}{5}$

$x = 15$

Let x be the length of the footbridge from C to D in m.

$\therefore$, the footbridge would be 15 m long

c) Let y be the length of the footbridge from C to E, in m.

$\frac{AC}{EC} = \frac{1}{5}$

$\frac{5}{y} = \frac{1}{5}$ $\rightarrow$ $y = 25$

$\therefore$ the footbridge from C to E would be 25 m long.