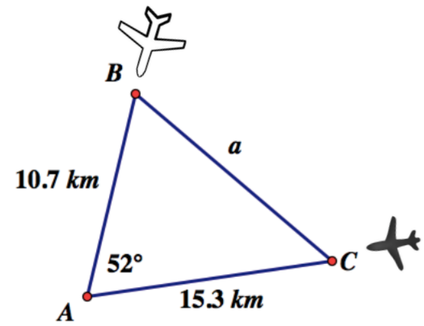
Developing the Cosine Law



Example 1

An air traffic controller at A is tracking two planes at B and C.

The planes are flying at the same altitude.

If the planes are less than 10 km apart, a warning must be issued to the pilots.

Should a warning be issued?

1) Hmm. This triangle has no 90° angle. So I cannot use the primary trig ratios.

2) What about the Sine Law?

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ Let's substitute.

$$\frac{a}{\sin 52^\circ} = \frac{15.3}{\sin B} = \frac{10.7}{\sin C}$$

Hmm. To find a... I could do:

$$\frac{a}{\sin 52^\circ} = \frac{15.3}{\sin B} \quad \text{OR} \quad \frac{a}{\sin 52^\circ} = \frac{10.7}{\sin C}$$

Problem. In neither situation I have two unknowns, which means I cannot solve this with Sine Law.

3) Apply the cosine Law:

$$a^2 = b^2 + c^2 - 2bc \cdot (\cos A)$$

$$a^2 = (15.3)^2 + (10.7)^2 - 2(15.3)(10.7)(\cos 52^\circ)$$

$$a^2 \doteq 234.09 + 114.49 - 201.58$$

$$a^2 \doteq 147$$

$$a^2 = 147$$

$$a \doteq 12.1$$

Now: $12.1 > 10$

$\therefore$, no warning needs to be issued since the planes are more than 10 km apart.

Example 2

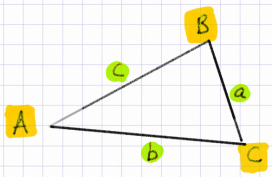
Recall how in our most recent class, we dropped a perpendicular from the vertex of an acute triangle to the opposite side.

By using the *sine ratio* to connect the height of the triangle to the two right triangles that were created, the *Sine Law* was developed.

Formally…

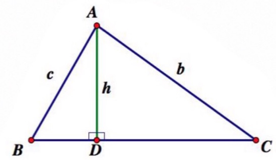
SINE LAW

Given an acute triangle ABC:



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{OR} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

NOTE: By convention, angles are denoted by uppercase letters.
Side lengths are denoted by lowercase letters.



$\sin C = \frac{h}{b}$

$b\left[\sin C = \frac{h}{\cancel{b}}\right]b$

$b \cdot \sin C = h$

$\sin B = \frac{h}{c}$

$c\left[\sin B\right] = \left[\frac{h}{\cancel{c}}\right]c$

$c \cdot \sin B = h$

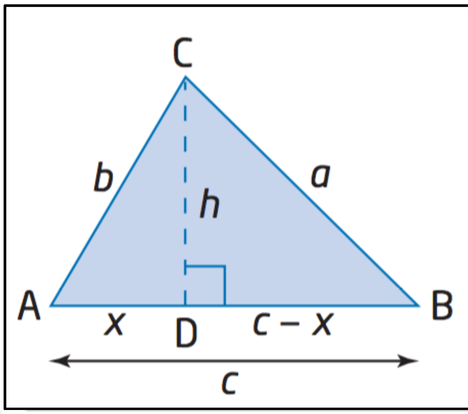
$b \cdot \sin C = c \cdot \sin B$

$\frac{\cancel{b} \cdot \sin C}{\cancel{b}c} = \frac{\cancel{c} \cdot \sin B}{b\cancel{c}}$

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

However…

… what if, after dropping the perpendicular, we had tried using the *cosine ratio* instead?



At left, consider $\triangle ADC$ and $\triangle BDC$.

Our goal is to develop a formula that only involves the measures of $\angle A$, $\angle B$, $\angle C$ and side lengths a, b, and c.

Let AD be x.
Then let BD be $c - x$

From $\triangle ADC$:

① By Pythagorean Theorem

$hyp^2 = leg_1^2 + leg_2^2$

$b^2 = x^2 + h^2$

② From the cosine ratio

$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$

$\cos A = \frac{x}{b}$

$b\left[\cos A\right] = \left[\frac{x}{\cancel{b}}\right]\cancel{b}$

$b(\cos A) = x$

From $\triangle BDC$:

③ By Pythagorean Theorem:

$hyp^2 = leg_1^2 + leg_2^2$

$a^2 = h^2 + (c-x)^2$ ⟵ Expand the final term

$a^2 = h^2 + (c-x)(c-x)$

$a^2 = h^2 + c^2 - cx - cx + x^2$

$a^2 = h^2 + c^2 - 2cx + x^2$ Re-arrange term order

$a^2 = x^2 + h^2 + c^2 - 2cx$ Now substitute. (see highlights)

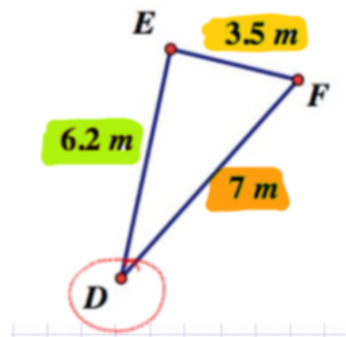
$a^2 = b^2 + c^2 - 2c[b(\cos A)]$ Expand.

$a^2 = b^2 + c^2 - 2cb(\cos A)$ Re-arrange

$$a^2 = b^2 + c^2 - 2bc(\cos A)$$

Example 3

What is the measure of angle D, to the nearest degree?



Hmm. Formula we have to use...

$d^2 = e^2 + f^2 - 2ef(\cos D)$

Let's re-arrange first!

$$d^2 = e^2 + f^2 - 2ef(\cos D)$$
$$-e^2 - f^2 \qquad -e^2 - f^2$$

$$\frac{d^2 - e^2 - f^2}{-2ef} = \frac{-2ef(\cos D)}{-2ef}$$

$$\frac{d^2 - e^2 - f^2}{-2ef} = \cos D$$

* This is an alternate formula that we can use to find angles with Cosine Law going forward!

* Fill in known values now.

* we can just start with this version of the formula in the future.

$$\frac{(3.5)^2 - (7)^2 - (6.2)^2}{-2(7)(6.2)} = \cos D$$

$$\frac{-75.19}{-86.8} = \cos D$$

$$0.866 \doteq \cos D$$

$$\cos^{-1}(0.866) = \angle D$$

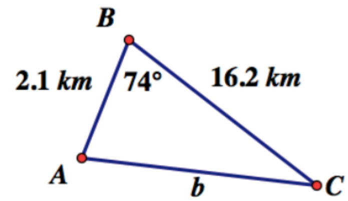
$$30 \doteq \angle D$$

*or look up ratio in cosine table

$\therefore$ angle D is about 30°

Opportunity to Learn

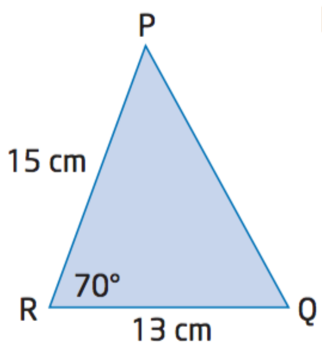
1. What is the length of b, to the nearest tenth of a kilometre?



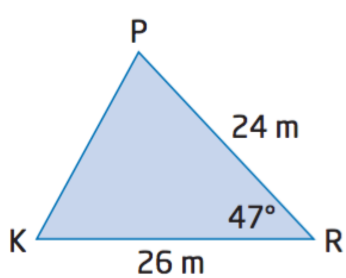
2. Solve each triangle. Round answers to the nearest unit, if necessary.

   NOTE: To solve a triangle means to find the measure of all sides and angles.

a)

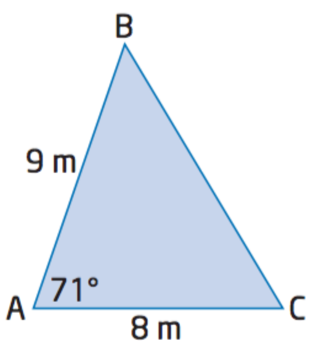


b)



2. (continued)

c)



3. Two observers who are 5 km apart simultaneously sight a small airplane flying between them.

   One observer measures a 51^{o} angle of elevation, while the other measures a 40.5^{o} angle of elevation.

   At what altitude is the airplane flying? Include a diagram with your solution.