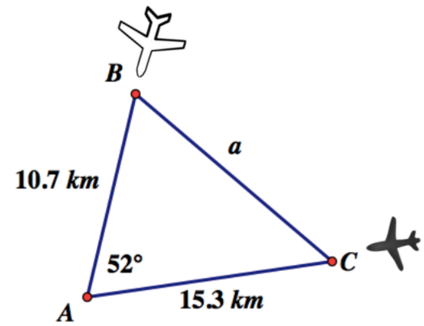
Developing the Cosine Law

Example 1



An air traffic controller at A is tracking two planes at B and C.

The planes are flying at the same altitude.

If the planes are less than 10 km apart, a warning must be issued to the pilots.

Should a warning be issued?

1) Hmm. This triangle has no 90° angle. So I cannot use the primary trig ratios.

2) What about the Sine Law?

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ Let's substitute.

$$\frac{a}{\sin 52°} = \frac{15.3}{\sin B} = \frac{10.7}{\sin C}$$

Hmm. To find a... I could do:

$$\frac{a}{\sin 52°} = \frac{15.3}{\sin B} \quad \text{OR} \quad \frac{a}{\sin 52°} = \frac{10.7}{\sin C}$$

Problem. Neither situation I have two unknowns. Which means I cannot solve this with Sine Law.

3) Apply the cosine law:

$$a^2 = b^2 + c^2 - 2bc \cdot (\cos A)$$
$$a^2 = (15.3)^2 + (10.7)^2 - 2(15.3)(10.7)(\cos 52°)$$
$$a^2 \doteq 234.09 + 114.49 - 201.58$$
$$a^2 = 147$$
$$a^2 = 147$$
$$a \doteq 12.1$$

Now: $12.1 > 10$

∴, no warning needs to be issued since the planes are more than 10 km apart.

Example 2

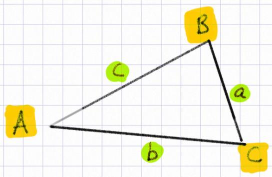
Recall how in our most recent class, we dropped a perpendicular from the vertex of an acute triangle to the opposite side.

By using the *sine ratio* to connect the height of the triangle to the two right triangles that were created, the *Sine Law* was developed.

Formally…

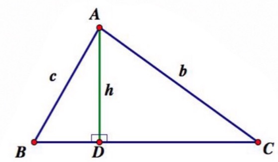
SINE LAW

Given an acute triangle ABC:



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{OR} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

NOTE: By convention, angles are denoted by uppercase letters. Side lengths are denoted by lowercase letters.



$\sin C = \frac{h}{b}$

$b\left[\sin C = \frac{h}{\cancel{b}}\right]\cancel{b}$

$b \cdot \sin C = \boxed{h}$

$\sin B = \frac{h}{c}$

$\left[\sin B\right] = \left[\frac{h}{\cancel{c}}\right]\cancel{c}$

$c \cdot \sin B = \boxed{h}$

$b \cdot \sin C = c \cdot \sin B$

$\frac{\cancel{b} \cdot \sin C}{\cancel{b}c} = \frac{\cancel{c} \cdot \sin B}{b\cancel{c}}$

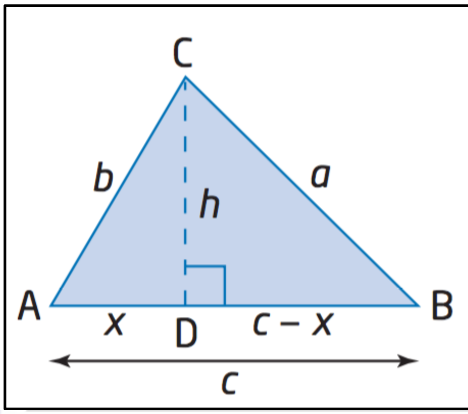
$$\boxed{\frac{\sin C}{c} = \frac{\sin B}{b}}$$

However…

… what if, after dropping the perpendicular, we had tried using the *cosine ratio* instead?

At left, consider $\triangle ADC$ and $\triangle BDC$.

Our goal is to develop a formula that only involves the measures of $\angle A$, $\angle B$, $\angle C$ and side lengths a, b, and c.



Let AD be x.
Then let BD be $c - x$

From $\triangle ADC$:

(1) By Pythagorean Theorem

$hyp^2 = leg_1^2 + leg_2^2$

$b^2 = x^2 + h^2$

(2) From the cosine ratio

$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$

$\cos A = \frac{x}{b}$

$b\left[\cos A\right] = \left[\frac{x}{\cancel{b}}\right]\cancel{b}$

$b(\cos A) = x$

From $\triangle BDC$:

③ By Pythagorean Theorem:

$$hyp^2 = leg_1^2 + leg_2^2$$

$$a^2 = h^2 + (c-x)^2 \leftarrow$$ Expand the final term

$$a^2 = h^2 + (c-x)(c-x)$$

$$a^2 = h^2 + c^2 - cx - cx + x^2$$

$$a^2 = h^2 + c^2 - 2cx + x^2$$ Re-arrange term order

$$a^2 = x^2 + h^2 + c^2 - 2cx$$ Now substitute. (see highlights)

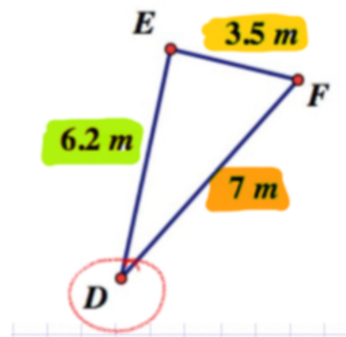
$$a^2 = b^2 + c^2 - 2c[b(\cos A)]$$ Expand.

$$a^2 = b^2 + c^2 - 2cb(\cos A)$$ Re-arrange

$$a^2 = b^2 + c^2 - 2bc(\cos A)$$

Example 3

What is the measure of angle D, to the nearest degree?



Hmm. Formula we have to use...

$d^2 = e^2 + f^2 - 2ef(\cos D)$

Let's re-arrange first!

$$d^2 = e^2 + f^2 - 2ef(\cos D)$$
$$-e^2 - f^2 \qquad -e^2 - f^2$$

$$\frac{d^2 - e^2 - f^2}{-2ef} = \frac{-2ef(\cos D)}{-2ef}$$

$$\frac{d^2 - e^2 - f^2}{-2ef} = \cos D$$

* This is an alternate formula that we can use to find angles with Cosine Law going forward!

* Fill in known values now.

$$\frac{(3.5)^2 - (7)^2 - (6.2)^2}{-2(7)(6.2)} = \cos D$$

* we can just start with this version of the formula in the future.

$$\frac{-75.19}{-86.8} = \cos D$$

$$0.866 \doteq \cos D$$

$$\cos^{-1}(0.866) = \angle D$$

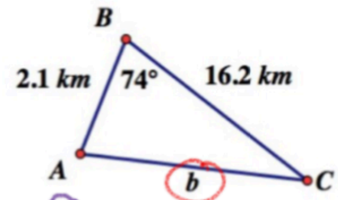
* or look up ratio in cosine table

$$30 \doteq \angle D$$

$\therefore$ angle D is about 30°

Opportunity to Learn

1. What is the length of b, to the nearest tenth of a kilometre?



$$b^2 = a^2 + c^2 - 2ac(\cos B)$$

① Fill in known values.

$$b^2 = (16.2)^2 + (2.1)^2 - 2(16.2)(2.1)(\cos 74^\circ)$$

$$b^2 \doteq 262.44 + 4.41 - 18.75$$

② Now with a calculator, suggest calculating in three steps as shown by highlighting

$$b^2 = 248.1$$

$$\sqrt{b^2} = \sqrt{248.1}$$

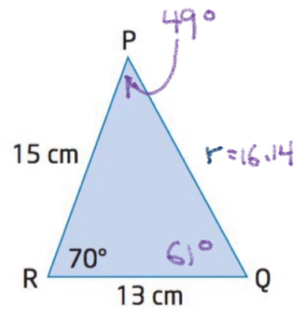
$$b \doteq 15.8$$

∴, b is about 15.8 km.

2. Solve each triangle. Round answers to the nearest unit, if necessary.

   NOTE: To solve a triangle means to find the measure of all sides and angles.

a)



*There are multiple ways or approaches for solving a triangle.

(1) Get r (using Cosine Law)

$r^2 = p^2 + q^2 - 2pq(\cos R)$

$r^2 = (13)^2 + (15)^2 - 2(13)(15)(\cos 70°)$

$r^2 \doteq 169 + 225 - 133.388$

$r^2 = 260.612$

$\sqrt{r^2} = \sqrt{260.612}$

$r \doteq 16.14$

(2) Get ∠Q (using alternate form of Cosine Law)

$\cos Q = \dfrac{q^2 - r^2 - p^2}{-2rp}$

$\cos Q = \dfrac{15^2 - (16.14)^2 - (13)^2}{-2(16.14)(13)}$

$\cos Q \doteq \dfrac{-204.5}{-419.6}$

$\cos Q \doteq 0.487$

$\angle Q = \cos^{-1}(0.487)$

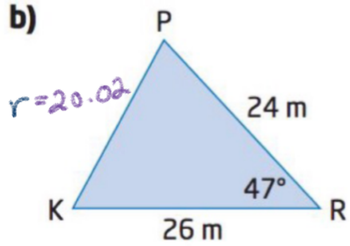
$\angle Q \doteq 61°$

(3) Get ∠P

$\angle P = 180° - 70° - 61°$

$\doteq 49°$

b)



*There are multiple ways or approaches for solving a triangle.

① Get r using the cosine law

$r^2 = k^2 + p^2 - 2kp(\cos R)$

$r^2 = (24)^2 + (26)^2 - 2(24)(26)(\cos 47°)$

$r^2 \doteq 576 + 676 - 851.134$

$r^2 = 400.866$

$\sqrt{r^2} = \sqrt{400.866}$

$r \doteq 20.02$

② Get ∠K using the Sine Law.

$\frac{\sin K}{k} = \frac{\sin R}{r}$

$\frac{\sin K}{24} = \frac{\sin 47°}{20.02}$

$24\left[\frac{\sin K}{\cancel{24}}\right] = \left[\frac{\sin 47°}{20.02}\right]24$

$\sin K \doteq 0.877$

$\angle K = \sin^{-1}(0.877)$

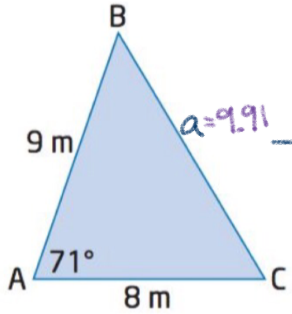
$\angle K \doteq 61°$

③ Get ∠P

$\angle P = 180° - 47° - 61°$

$= 72°$

c)



*There are multiple ways or approaches for solving a triangle.

① Get a using the Cosine Law.

$a^2 = b^2 + c^2 - 2bc(\cos A)$

$a^2 = (8)^2 + (9)^2 - 2(8)(9)(\cos 71°)$

$a^2 = 64 + 81 - 46.882$

$a^2 = 98.118$

$\sqrt{a^2} = \sqrt{98.118}$

$a \doteq 9.91$

② Get ∠B using Sine Law.

$\frac{\sin B}{b} = \frac{\sin A}{a}$

$\frac{\sin B}{8} = \frac{\sin 71°}{9.91}$

$\cancel{8}\left[\frac{\sin B}{\cancel{8}}\right] = \left[\frac{\sin 71°}{9.91}\right]8$

$\sin B \doteq 0.763$

$\angle B = \sin^{-1}(0.763)$

$\angle B \doteq 50°$

③ Get ∠C

$\angle C = 180° - 71° - 50°$

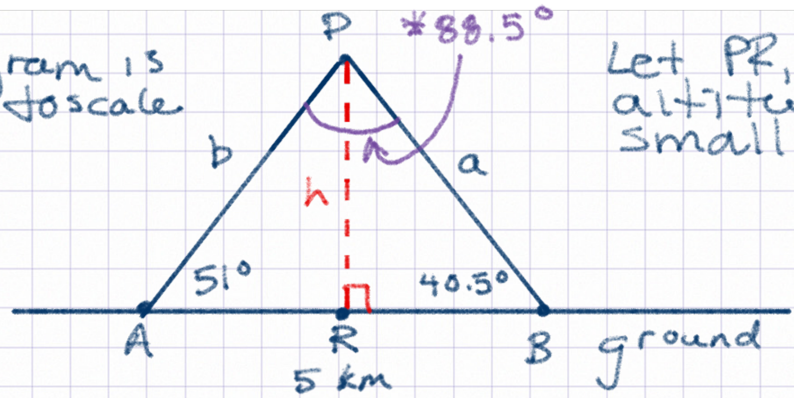
$= 59°$

3. Two observers who are 5 km apart simultaneously sight a small airplane flying between them.

   One observer measures a 51° angle of elevation, while the other measures a 40.5° angle of elevation.

   At what altitude is the airplane flying? Include a diagram with your solution.

*diagram is not to scale



Let PR, or h, be the altitude of the small airplane.

To obtain h, we first need either a or b.

To obtain a or b, we first need ∠P.

① Get ∠P.

$\angle P = 180° - 51° - 40.5°$
$= 88.5°$ *

② Get b

$\frac{b}{\sin B} = \frac{p}{\sin P}$

$\frac{b}{\sin 40.5°} = \frac{5}{\sin 88.5}$

$\cancel{\sin 40.5}\left[\frac{b}{\cancel{\sin 40.5°}}\right] = \left[\frac{5}{\sin 88.5}\right]\sin 40.5$

$b \doteq 3.248$

③ Get h

$\sin A = \frac{opp}{hyp}$

$\sin 51° \doteq \frac{h}{3.248}$

$3.248\left[\sin 51°\right] \doteq \left[\frac{h}{\cancel{3.248}}\right]\cancel{3.248}$

$2.5 \doteq h$

∴ the altitude of the small airplane is about 2.5 km.