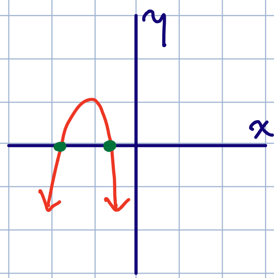
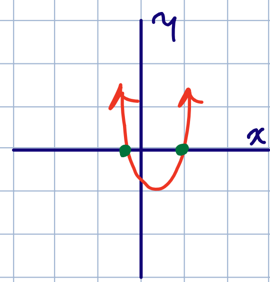
Deriving the Quadratic Formula

Background

If a quadratic relation crosses the x-axis, it will have two intercepts, for example:

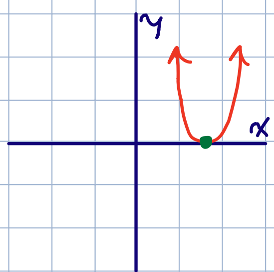


or

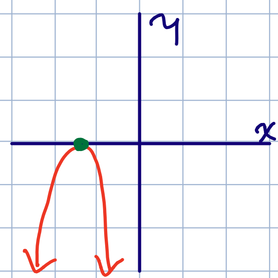


etc.

The vertex of a quadratic relation could rest on the x-axis, meaning there is one (really, two equal) x-intercepts, for example:

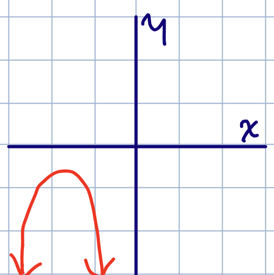


or

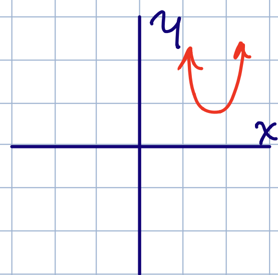


etc.

Or, a quadratic relation may not intersect the x-axis at all, for example:



or



etc.

Given a quadratic relation in standard form:

$$y = ax^2 + bx + c$$

...and using the "quadratic formula":

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

... you can calculate the x-intercepts (if they exist). But where does the quadratic formula "come from"? Let's take a look.

Deriving the Quadratic Formula

Given any quadratic relation:

$$y = ax^2 + bx + c$$

or

$$ax^2 + bx + c = y$$

...our first step is to set $y=0$ (since we want to find the x-intercepts)

$$ax^2 + bx + c = 0$$

... we want to isolate x, so, divide both sides by a.

$$\frac{ax^2 + bx + c}{a} = \frac{0}{a}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

... now, to continue isolating x, we will eliminate the constant term on the left side.

$$x^2 + \frac{b}{a}x + \frac{c}{a} - \frac{c}{a} = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

... now we will complete the square on the left side.

$$\underline{\underline{x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}}} - \frac{b^2}{4a^2} = -\frac{c}{a}$$

Side calculation:

(divide coefficient of middle term by 2, then square it)

$$\left(\frac{\frac{b}{a}}{2}\right)^2 = \left(\frac{b}{a} \cdot \frac{1}{2}\right)^2$$

$$= \left(\frac{b}{2a}\right)^2$$

$$= \frac{b^2}{4a^2}$$

Now, we will factor the perfect square trinomial:

$$\underline{\underline{\left(x + \frac{b}{2a}\right)^2}} - \frac{b^2}{4a^2} = -\frac{c}{a}$$

OK, now let's simplify on the left side:

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

Whew! Remember, we want to isolate x.

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}\cdot\frac{4a}{4a}$$

Let's first combine the terms on the right side by adding them. We need common denominators.

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

OK. Next step... remember... isolate x!

$$\pm\sqrt{\left(x+\frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

Let's take the square root of both sides.

$$x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

We can simplify the denominator. (square root of 4 is 2, square root of a^2 is a.)

$$x+\frac{b}{2a} - \frac{b}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

Now, continue isolating x by subtracting the second term on both sides...

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Finally, just add the two terms on the right side, since they already share a common denominator.

And there you have it! The quadratic formula is not "magic", just the result of a lot of careful algebraic work.

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$